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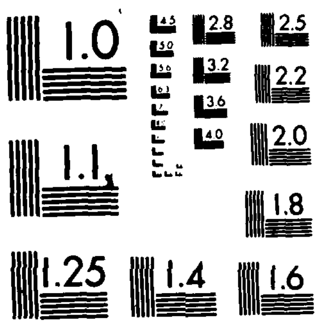

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ON THE TAYLOR APPROXIMATION OF CONTROL SYSTEMS

Alberto Bressan

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ON THE TAYLOR APPROXIMATION OF CONTROL SYSTEMS

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ABSTRACT

Let g_i $(i = 1, \ldots, m)$ be smooth vector fields on R^d, and let $T^{n-1} g_i$ be their Taylor expansions of order $n - 1$ at the origin. The system $\dot{x}(t) = \sum_{i=1}^{m} g_i(x(t))u_i(t)$, $x(o) = 0 \in R^d$ generates an input-output map $u(\cdot) \to x(u,\cdot)$ whose n-th order Taylor approximation $x_n(u,\cdot)$ can be obtained by computing the n-th Picard iterate for the reduced system $\dot{x}(t) = \sum_{i=1}^{m} (T^n g_i)(x(t))\, u_i(t)$, $x(o) = 0$, discarding the terms of order $> n$. For $z \in R^d$, directional error bounds of the form

$$| < z, x(u,t) - x_n(u,t) > | \leq C\, t^{n-p} \int_0^t | \int_0^s \sum_{i=1}^{m} g_i(o)u_i(\sigma)\, d\sigma |^p ds$$

can be given. These estimates improve those supplied by the classical Taylor's theorem and yield results concerning local non-controllability.

AMS (MOS) Subject Classifications: 93C10, 49E15

Key Words: Nonlinear control systems, Picard iterates, Taylor expansion.

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Istituto di Matematica Applicata, Università di Padova, 35100 ITALY

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SIGNIFICANCE AND EXPLANATION

Consider the control system on $\mathbb{R}^d$:

$$\frac{d}{dt} x(t) = \sum_{i=1}^{m} g_i(x(t))\cdot u_i(t), \quad x(o) = 0.$$

Given a control u, in general there exists no explicit formula to exactly compute the corresponding trajectory $t \to x(u,t)$. ~~The present~~ paper provides a simple method of constructing approximations $x_n(u,t)$ of $x(u,t)$, which are linear combinations of integrals of the controls u_i and which can be explicitly computed. The coefficients of these linear combinations are obtained from the partial derivatives of the vector fields g_i at the origin, using a Picard iterative procedure.

Here the approximation error is the vector $\varepsilon_n(u,t) = x_n(u,t) - x(u,t)$. New and precise bounds on the length of $\varepsilon_n(u,t)$, as well as on the size of the projections of $\varepsilon_n(u,t)$ on some special subspaces of $\mathbb{R}^d$ are given.

The present formula can be used for a systematic study of a control system over a short interval of time, in much the same way as the classical Taylor expansion is used to study the local properties of differentiable functions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE TAYLOR APPROXIMATION OF CONTROL SYSTEMS

Alberto Bressan

Consider an autonomous nonlinear system on $\mathbb{R}^d$ with control entering linearly:

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))\, u_i(t)$$
$$x(o) = 0, \quad t > 0. \tag{1.1}$$

Systems of the form (1.1) were first studied in a paper by Hermes and Haynes [4] and received continued attention since then. If f and the g_i's are smooth vector fields, then (1.1) yields a smooth input-output map $\phi: L^1([0,\infty); \mathbb{R}^m) \to C([0,\infty); \mathbb{R}^d)$ defined by $\phi(u)(t) = x(u,t)$, where $x(u,\cdot)$ is the unique solution of (1.1) corresponding to the control u. In general there exist no explicit formulas giving the trajectory $x(u,\cdot)$ directly in terms of the control. It is therefore natural to look for some computable approximation of the map ϕ. The Taylor expansion of ϕ in terms of Volterra kernels was considered by Brockett [1]. The local aproximation of a control system of the form

$$\dot{x}(t) = \sum_{i=1}^{m} g_i(x(t))u_i(t), \quad x(o) = 0 \tag{1.2}$$

by means of an auxiliary (nilpotent) system is studied in [5]. For analytic systems, expansions in formal power series are given in [3].

In the present paper we approximate the input-output may ϕ generated by (1.2) with linear combinations of certain iterated integrals, here called integral monomials. Using functional analytic techniques, we derive a simple procedure to recursively compute the coefficients of the Taylor expansion for ϕ in terms of the Taylor coefficients of the g_i's at the origin. No analyticity assumptions are needed. Our main concern is a

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precise estimate of the approximation errors. Since the system (1.2) is linear in u, these errors are essentially due to the nonlinearity of the vector fields g_i, which depend only on x. We thus obtain bounds which depend not on the size of the control $u(\cdot)$ but only on the norm of a first order approximation to the trajectory $x(u,\cdot)$. The estimates contained in Theorems 1 and 2, in sections 5 and 6 respectively, improve those supplied by the classical Taylor's theorem and are sharp enough to yield results on local non-controllability for the system (1.1). Examples are provided in sections 4 to 6. Some related results recently appeared in [6].

§ 2. Preliminary Abstract Results

Given two Banach spaces E and F, $k > 0$, we denote by $L^k(E,F)$ the space of continuous k-linear mappings Λ from $\bigotimes_k E = E\times E\times\ldots\times E$ (k times) into F, with the operator norm

$$\|\Lambda\| = \sup \{\|\Lambda(v_1, \ldots, v_k)\|_F;\ \|v_i\|_E \leq 1,\ i = 1, \ldots, k\}.$$

If $\phi: E \to F$ is a smooth mapping, its k-th Fréchet derivative at a point $u \in E$ is $D^k\phi(u) \in L^k(E,F)$. It is well known that high-order derivatives are symmetric multilinear mappings. $D^k\phi(u)$ is therefore completely determined by assigning its values on elements of the form $v^{[k]} = (v,v,\ldots,v) \in \bigotimes_k E$. We write B_ρ for the closed ball centered at the origin with radius ρ, and $T^n\phi$ for the n-th order Taylor expansion of the map ϕ at the origin, i.e.

$$(T^n\phi)(u) = \sum_{i=0}^{n} \frac{1}{i!} D^i\phi(o)\cdot u^{[i]}.$$

Given a function $\Phi = \Phi(u,x)$ defined on a product space $E\times F$, its partial derivatives are denoted by ∂_u, ∂_x. In this case, $T^n\Phi$ stands for the n-th order Taylor expansion of Φ at $(0,0)$. The Landau order symbols O and o will also be used. For the basic properties of differential calculus in Banach spaces, our general reference is Dieudonné [2]. The approximation procedure considered in this paper is based on the following simple result:

Proposition 1. Let E,F be Banach spaces, let $(u,x) \to \Phi(u,x)$ be a C^k map from a neighborhood of the origin in $E\times F$ into F such that $\Phi(0,0) = 0$, $\partial_x\Phi(0,0) = 0$, and let the map $u : E \to F$ be implicitly defined by $\phi(u) = \Phi(u,\phi(u))$. If Ψ,ψ are C^k maps such that $T^n\Psi = T^n\Phi$ and $T^{n-1}\psi = T^{n-1}\phi$ for some n, $0 < n < k$, then the n-th order Taylor expansion at the origin of the map $u \to \Psi(u,\psi(u))$ coincides with $T^n\phi$.

Proof. Since the maps $\phi(\cdot)$ and $\Psi(\cdot,\psi(\cdot))$ are both C^k, we only need to show that, under the above hypothesis,

$$\lim_{n\to 0} \|\phi(u) - \Psi(u,\psi(u))\|\cdot\|u\|^{-n} = 0. \tag{2.1}$$

By Taylor's theorem, for a suitable constant C and for u,x sufficiently close to the origin one has

$$\|\phi(u)\| < C\|u\|,\ \|\psi(u)\| < C\|u\|,\ \|\phi(u) - \psi(u)\| < C\|u\|^n,$$

$$\|\partial_x\Psi(u,x)\| < C(\|u\|+\|x\|),\ \|\Phi(u,x) - \Psi(u,x)\| < C\ (\|u\|^{n+1} + \|x\|^{n+1}).$$

These inequalities imply

$$\|\phi(u) - \Psi(u,\psi(u))\| < \|\Phi(u,\phi(u)) - \Psi(u,\phi(u))\| + \|\Psi(u,\phi(u)) - \Psi(u,\psi(u))\|$$

$$< C(\|u\|^{n+1}+\|\phi(u)\|^{n+1}) + \int_0^1 \|\partial_x\Psi(u,\xi\phi(u)+(1-\xi)\psi(u))\|\cdot\|\phi(u) - \psi(u)\|d\xi$$

$$< C(1+C^{n+1})\|u\|^{n+1} + C(\|u\| + C\|u\|)\cdot C\|u\|^n,$$

which proves (2.1).

Corollary 1. If Φ, ϕ satisfy the assumptions made in Proposition 1, then

$$\lim_{u\to 0} \|\ \Phi(u,(T^n\phi)(u)) - (T^n\phi)(u)\ \|\cdot\|u\|^{-n} = 0. \tag{2.2}$$

§3 The Taylor Formula

Most of this paper is concerned with the system (1.2). Notice that, by setting $u_1 \equiv 1$, a control system of the form (1.1) is recovered as a special case of (1.2). To simplify all further discussion, we assume that the vector fields g_i are C^∞. In the following, $|g_i(x)|$ denotes the euclidean norm of the vector

$g_i(x) \in \mathbb{R}^d$, while $|\nabla g_i(x)|$ is the operator norm of the $d \times d$ matrix of first order partial derivatives of g_i at x. The set of admissible controls is defined as

$U = \{u = (u_1, \ldots, u_m) \in L^1([0,\infty);\mathbb{R}^m);\ |u_i(t)| \leq 1,\ i = 1, \ldots, m,\ t \geq 0\}$.

The L^1 norm on the set of controls and the C^0 norm on the set of trajectories will be always used. We write (1.2) in the more compact form

$$\dot{x}(t) = G(x(t))u(t) \quad , x(o) = 0. \tag{3.1}$$

G is therefore a C^∞ map from $\mathbb{R}^d$ into $L(\mathbb{R}^m, \mathbb{R}^d)$. Call $G^{(j)}(z)$ its j-th derivative at $z \in \mathbb{R}^d$ and let $G_n = T^n G$. Set $E = L^1([0,\infty);\mathbb{R}^m)$, $F = C([0,\infty);\mathbb{R}^d)$, and define the map $\Phi : E \times F \to F$ by

$$\Phi(u,y)(t) = \int_0^t G(y(s))u(s)\, ds \ . \tag{3.2}$$

Then $\Phi = \Phi'' \circ \Phi'$, with $\Phi'(u,y)(t) = (u(t), G(y(t)))$

and

$$\Phi''(u,Z)(t) = \int_0^t Z(s)u(s)ds.$$

Φ' is a C^∞ substitution operator and Φ'' is bilinear. Hence Φ is k times continuously Fréchet differentiable, for all k. In particular, $\partial_x^j \Phi(u_o,x_o)(t)$ is the j-linear map

$$y^{[j]} \to \int_0^t G^{(j)}(x_o(s))\, y^{[j]}(s)\, u_o(s)\, ds, \tag{3.3}$$

$\partial_u \partial_x^j \Phi(u_o,x_o)(t)$ is the multilinear map

$$(u,y^{[j]}) \to \int_0^t G^{(j)}(x_o(s))\, y^{[j]}(s)\, u(s)\, ds \tag{3.4}$$

and $\partial_u^i \partial_x^j \Phi \equiv 0$ for $i > 1$, because the dependence on u is linear. Notice that the input-output map $u(\cdot) \to x(u,\cdot) = \phi(u)(\cdot)$ generated by (1.2), or equivalently by (3.1), is implicitly defined by the equation $\phi(u) = \Phi(u,\phi(u))$, and that both Φ and $\partial_x \Phi$ vanish at $(0,0)$. The Taylor expansion of Φ at the origin can therefore by computed recursively, by means of Proposition 1. We will show that $T^n \phi$ can always be written as a finite sum of certain iterated integrals of the control u, here called integral monomials (integromials, in short).

Definition. The family $M = M(m,d)$ of integral monomials is the smallest set of mappings μ from $L^1([0,\infty);\mathbb{R}^m)$ into $\mathbb{R}^d$ with the following properties:

i) For every linear $A : \mathbb{R}^m \to \mathbb{R}^d$, the map

$$\mu : (u,t) \to \int_0^t A \cdot u(s)ds \tag{3.5}$$

is in $\mathcal{M}$

ii) If $\mu_1, \ldots, \mu_k \in \mathcal{M}$ and if $B:(\mathbb{R}^d)^k \to L(\mathbb{R}^m;\mathbb{R}^d)$ is k-linear, then $\mathcal{M}$ also contains the map

$$\mu : (u,t) \to \int_0^t B(\mu_1(u,s), \ldots, \mu_k(u,s))\, u(s)ds. \tag{3.6}$$

Using a canonical identification, we shall regard integral monomials as multilinear mappings from $L^1([0,\infty);\mathbb{R}^m)$ into $C^0([0,\infty);\mathbb{R}^d)$. If $\mu \in \mathcal{M}$ is k-linear, we say that μ has order k. Notice that if in (3.6) μ_i has order $\nu_i (i = 1, \ldots, k)$, then μ has order $1 + \nu_1 + \ldots + \nu_k$. Consider now the approximate system

$$\dot{x}(t) = G_\nu(x(t))u(t) \;, \quad x(0) = 0, \tag{3.7}$$

recalling our definition : $G_\nu = T^\nu G$. The first Picard iterate for (3.2)

$$P_1(u,t) = \int_0^t G(0)u(s)ds \tag{3.8}$$

is an integral monomial of the form (3.5). Moreover, if the n-th Picard iterate P_n can be written as a finite sume of integral monomials, say

$$P_n(u,t) = \sum_{j=1}^{N(n)} \mu_j(u,t),$$

then the same holds for P_{n+1}. Indeed

$$P_{n+1}(u,t) = \int_0^t \sum_{k=0}^{\nu} \frac{1}{k!} G^{(k)}(0)(P_n(u,s))^{[k]}u(s)ds$$

can be reexpanded into a finite sum of integral monomials of the form (3.6), namely

$$P_{n+1}(u,t) = \sum_{k=0}^{\nu} \sum_{\sigma\in\Gamma} \int_0^t \frac{1}{k!} G^{(k)}(0)\ (\mu_{\sigma(1)}(u,s),\ldots,\mu_{\sigma(k)}(u,s))u(s)ds, \tag{3.9}$$

where $\Gamma = \Gamma(k,N(n))$ is the set of all mappings σ from $\{1,2,\ldots,k\}$ into $\{1,\ldots,N(n)\}$.

In practice, the k-th order Taylor expansion $T^k\phi$ at the origin for the input-output map $\phi: u(\cdot) \to x(u,\cdot)$ generated by (3.1) can be obtained by either of the following methods:

I) Compute the k-th Picard iterate for the system

$$\dot{x}(t) = G_{k-1}(x(t))u(t) \quad , \; x(o) = 0 \tag{3.10}$$

and discard the integral monomials of order $> k$.

II) Set $x_n(u,\cdot) = (T^n\phi)(u)(\cdot)$ and recursively derive x_n from x_{n-1} $(n = 1,\ldots,k)$ by expanding the map

$$(u,t) \to \int_o^t G_{n-1}(x_{n-1}(u,s))\, u(s)ds \tag{3.11}$$

into a sum of integral monomials, discarding those which have order $> n$.

Indeed, for $n \geq 1$ set

$$\Phi_n(u,x)(t) = \int_o^t G_{n-1}(x(s))\, u(s)ds \tag{3.12}$$

and check that the Taylor expansions at (0,0) of Φ_n and Φ (defined at (3.2)) coincide up to order n. Let P_n be the n-th Picard iterate for (3.10), regarded as a map $u(\cdot) \to P_n(u,\cdot)$ from L^1 into C^o. Assume that $T^{n-1}P_{n-1} = T^{n-1}\phi$. Then, by setting $\Psi = \Phi_k$, $\psi = P_{n-1}$, Proposition 1 yields $T^nP_n = T^n\phi$, provided $n \leq k$. By induction, $T^kP_k = T^k\phi$. Observing that P_k is a finite sum of integromials, to obtain its k-th order Taylor expansion at the origin one merely discards the terms of order $> k$. This justifies I). Now set $\Psi = \Phi_n$, $\psi = T^{n-1}\phi$. By Proposition 1 the map $u \to \Phi_n(u,(T^{n-1}\phi)(u))$, otherwise defined at (3.11), has the same n-th order Taylor expansion at the origin as ϕ. Discarding from (3.11) the integral monomials of order $> n$ we thus recover $T^n\phi$ from $T^{n-1}\phi$. One obtains $T^k\phi$ by repeating the above procedure for $n = 1, \ldots, k$.

§4 Examples

Example 1. The third order Taylor expansion for the scalar system

$$\dot{x}(t) = \cos x(t)u_1(t) + u_2(t), \qquad x(o) = 0$$

is

$$(T^3\phi)(u)(t) = \int_o^t [u_1(s) + u_2(s)]ds - \tfrac{1}{2}\int_o^t u_1(\sigma_1)\Big(\int_o^{\sigma_1} u_1(\sigma_2)d\sigma_2\Big)^2 d\sigma_1$$

$$- \int_o^t u_1(\sigma_1)\Big(\int_o^{\sigma_1} u_1(\sigma_2)d\sigma_2\Big)\Big(\int_o^{\sigma_1} u_2(\sigma_2)d\sigma_2\Big)d\sigma_1 - \tfrac{1}{2}\int_o^t u_1(\sigma_1)\Big(\int_o^{\sigma_1} u_2(\sigma_2)d\sigma_2\Big)^2 d\sigma_1.$$

If u_1 is constrained to be identically 1 we obtain an approximation for the system $\dot{x}(t) = \cos x(t) + u(t)$, $x(0) = 0$, namely

$$x(u,t) = t - \frac{t^3}{6} + \tfrac{1}{2}\int_0^t u(\sigma)d\sigma - \tfrac{1}{2}\int_0^t \left(\int_0^{\sigma_1} u(\sigma_2)d\sigma_2\right)^2 d\sigma_1$$

$$- \int_0^t \sigma_1 \int_0^{\sigma_1} u(\sigma_2)d\sigma_2 d\sigma_1 + 0(t^4).$$

Example 2. Consider on $\mathbb{R}^3$ the control system

$$\dot{x} = X(x) + Y(x)\cdot u \quad , x(0) = 0. \tag{4.1}$$

Assume that $X(o) = 0$ and span $\{(ad^{\nu}X,Y);\ \nu = 0,\ 1,\ 2\} = \mathbb{R}^3$. Then the map $\pi : (s_1, s_2, s_3) \to (\exp S_1\ Y)\cdot(\exp s_2[Y,X])\cdot(\exp s_3[[Y,X],X])\ (o)$ defines a local chart of a neighborhood of the origin in $\mathbb{R}^3$. In this chart, the third order Taylor approximation for (4.1) takes the form

$$x_1(u,t) = \int_0^t u(s)ds + \frac{k_1}{2} \int_0^t \left(\int_0^{\sigma_1} u(\sigma_2)d\sigma_2\right)^2 d\sigma_1 + 0(t^4)$$

$$x_2(u,t) = \int_0^t \int_0^{\sigma_1} u(\sigma_2)d\sigma_2 d\sigma_1 + \frac{k_2}{2} \int_0^t \left(\int_0^{\sigma_1} u(\sigma_2)d\sigma_2\right)^2 d\sigma_1 + 0(t^4) \tag{4.2}$$

$$x_3(u,t) = \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} u(\sigma_3)d\sigma_3\ d\sigma_2 d\sigma_1 + \frac{k_3}{2}\int_0^t \left(\int_0^{\sigma_1} u(\sigma_2)d\sigma_2\right)^2 d\sigma_1 + 0(t^4),$$

where the constants k_1, k_2, k_3 are obtained from the linear relation

$k_1Y(o) + k_2[Y,X](o) + k_3[[Y,X],X](o) = [Y,[Y,X]](o)$.

To prove (4.2), we need to compute some Taylor coefficients of X and Y in the given chart. The definition of π implies that $Y \equiv (1,0,0)$, $[Y,X](o) = (0,1,0)$ and $[[Y,X],X](o) = (0,0,1)$. We thus have

$$\frac{\partial X}{\partial s_1}(o) = Y\cdot X(o) = [Y,X](o) = (0,1,0)$$

$$\frac{\partial X}{\partial s_2}(o) = [Y,X]\cdot X(o) = [[Y,X],X](o) = (0,0,1)$$

$$\frac{\partial^2 x}{\partial s_1^2}(o) = Y(Y \cdot X)(o) = [Y,[Y,X]](0) = (k_1, k_2, k_3).$$

One can now multiply the vector field X by a fictitious control $u_o \equiv 1$ and use Theorem 1 to get (4.2).

Remark: It is clear that the truncated Taylor expansions are not invariant un . change of coordinates. The above example shows how the local Lie algebraic structure the system can yield a chart in which the expansion takes the simplest possible f For a general procedure to construct "canonical" charts, see [5 p. 127].

§5 Error bounds.

High-order Taylor expansions are the primary tool in the local study of functionals at points of singularity. If the origin is a stationary point for a map $\Lambda : E \to \mathbb{R}$, one classically proves that Λ attains a local minimum at 0 by showing that for some $n > 1$

$$(T^n\Lambda)(u) - \Lambda(o) > C \cdot \|u\|^n \tag{5.1}$$

for all u is a neighborhood of the origin. One would hope to use a similiar argument and prove non-controllability results concerning the system (1.1). For example, let $f(o) = 0$. Then (1.1) is not locally controllable at the origin if, for some nontrivial $z^* \in \mathbb{R}^d$, $T > 0$, the functional $\Lambda(u) = \langle z^*, x(u,T)\rangle$ attains a local minimum at $\bar{u} \equiv 0$. Coercivity conditions such as (5.1), however, can seldom be obtained in connection with control systems. To overcome this difficulty, one needs bounds on the error $\varepsilon(u) = \|(T^n\phi)(u) - \phi(u)\|$ which are sharper then the classical bound $\varepsilon(u) < C\|u\|^{n+1}$ supplied by Taylor's theorem.

Lemma 1. Let $x(u,\cdot)$ be the solution of (1.2) corresponding to the control u. Then there exists a $t_o > 0$ such that, for every continuous map y from $[0,t_o]$ into the unit ball $B_1 \subseteq \mathbb{R}^d$,

$$|y(t) - x(u,t)| < 2 \sup \{ | \int_0^\tau G(y(s))u(s)ds - y(\tau)|;\ 0 < \tau < t\} \tag{5.2}$$

for all $t \in [0,t_o]$, $u \in U$.

Proof. Set $t_o = M^{-1}$, with

$M = 2m \cdot \sup \{|g_i(x)| + |\nabla g_i(x)|,\ x \in B_1,\ i = 1, \ldots, m\}$.

Then the map Φ defined at (3.2) satisfies

$\| \Phi(u,y_1) - \Phi(u,y_2)\|_{C[0,t_o]} < \frac{1}{2}\|y_1 - y_2\|_{C[0,t_o]}$

and $\Phi(u,y_1)(t) \in B_1$, for every $u \in U$, $t \in [0,t_o]$ and every continuous maps $y_1, y_2 : [0,t_o] \to B_1$. Since $x(u,\cdot)$ *is a fixed point for the may* $y \to \Phi(u,y)$, the contraction mapping theorem yields $\|y - x(u,\cdot)\| < 2\|y - \Phi(u,y)\|$, i.e., (5.2).

Lemma 2. For each $u \in U$, let $x(u,\cdot)$ be the solution of (3.1) and let $y_n(u,\cdot)$ be the solution of

$$\dot{y}(t) = G_{n-1}(y(t))u(t), \quad x(o) = 0. \tag{5.3}$$

Then there exist C and $t_o > 0$ such that

$$|y_n(u,t) - x(u,t)| < C \int_0^t | \int_0^s G(o)u(\sigma)d\sigma|^n ds \tag{5.4}$$

for all $t \in [0,t_o]$, $u \in U$.

Proof. Choose $t_1 > 0$ so small that $y_n(u,t) \in B_1$ for all $u \in U$, $t \in [0,t_1]$. Lemma 1 implies the existence of a $t_o > 0$ such that $t_o < t_1$ and

$$|y_n(u,t) - x(u,t)| < 2 \sup \{ | \int_0^\tau G(y_n(u,s))u(s)ds - y_n(u,\tau)|,\ 0 < \tau < t\} \tag{5.5}$$

for all $t \in [0,t_o]$, $u \in U$. Using (5.3) we have

$$\begin{aligned} |y_n(u,t) - x(u,t)| &< 2 \sup \{ | \int_0^\tau [G(y_n(u,s)) - G_{n-1}(y_n(u,s))]u(s)ds|,\ 0 < \tau < t\} \\ &< 2 \cdot C_1 \int_0^t |y_n(u,s)|^n ds, \end{aligned} \tag{5.6}$$

where the constant C_1 depends only on the size of the n-th derivative of G on B_1.

Set

$M = m \cdot \sup \{ |\nabla(T^{n-1}g_i)(z)|;\ z \in B_1 \subset \mathbb{R}^d,\ i = 1, \ldots, m\}$,

$\eta(t) = | \int_0^t G(o)u(s)ds|$, $w(t) = |y_n(u,t) - \int_0^t G(o)u(s)ds|$. For almost every $t \in [0,t_o]$, $\dot{w}(t) < | G_{n-1}(y_n(u,t)) - G_{n-1}(o) | < M \cdot |y_n(u,t)| < M(w(t) + \eta(t))$.

Since $w(o) = 0$, Gronwall's lemma yields $w(t) \leq C_2 \int_o^t \eta(s)ds$, with $C_2 = Me^{Mt_o}$. Using Hölder's inequality we get the estimate

$$2 \int_o^t |y(u,s)|^n ds \leq 2 \int_o^t [\eta(\tau) + C_2 \int_o^\tau \eta(s)ds]^n d\tau$$

$$\leq \int_o^t (2\eta(\tau))^n d\tau + \int_o^t (2C_2 \int_o^\tau \eta(s)ds)^n d\tau \tag{5.7}$$

$$\leq \int_o^t (2\eta(\tau))^n d\tau + (2C_2)^n \cdot t \ (\int_o^t \eta^n(s)ds)t^{n-1} \leq C_3 \int_o^t \eta^n(\tau)d\tau,$$

with $C_3 = 2^n + (2C_2\ t_o)^n$. Combining (5.6) with (5.7) one has (5.3), with $C = C_1C_3$.
The above lemmas provide a simple estimate for the error in the Taylor expansion of ϕ.
Theorem 1. Let $x(u,\cdot)$ be the solution of (3.1). Set $G_{n-1} = T^{n-1}G$, $x_n(u,t) = (T^n\phi)(u)(t)$. Then there exist $C, t_o > 0$ such that

$$|x(u,t) - x_n(u,t)\ | \leq C \int_o^t |\int_o^s G(o)u(\sigma)\ d\sigma|^n ds +$$
$$+ 2 \sup \{ |x_n(u,t) - \int_o^\tau G_{n-1}(x_n(u,s))u(s)ds|;\ 0 \leq \tau \leq t\}, \tag{5.8}$$

for every $u \in U$, $t \in [0,t_o]$.

Indeed, $|x(u,t) - x_n(u,t)| \leq |x(u,t) - y_n(u,t)| + |y_n(u,t) - x_n(u,t)|$. The estimates (5.4) and (5.2) with G replaced by G_{n-1} yield (5.8).
Example 3. Consider the bidimensional system $\frac{d}{dt}(x,y) = (u+y\cdot\cos x,\ x^2-x^3)$, $(x(0),y(o)) = (0,0)$. Its third order approximation is
$x(u,t) = \int_o^t u(s)ds + \varepsilon_1(u,t)$, $y(u,t) = \int_o^t (\int_o^s u(\sigma)d\sigma)^2 ds + \varepsilon_2(u,t)$.
Theorem 2 yields the bounds

$$|\varepsilon_1(u,t)| \leq C \int_o^t |\int_o^s u(\sigma)d\sigma|^3 ds + 2 \int_o^t \int_o^{\sigma_1} (\int_o^{\sigma_2} u(\sigma_3)d\sigma_3)^2 d\sigma_2 d\sigma_1, \tag{5.9}$$

for $i = 1, 2$, $u \in U$ and t sufficiently small. From (5.9) we see that $|\varepsilon_2(u,t)| = o(\int_0^t (\int_0^s u(\sigma)d\sigma)^2 ds)$, hence, for small t, $x(u,t) > 0$ and the system is not locally controllable. For any fixed $t_o > 0$, consider the control $u_\lambda(t) = \cos \lambda t$. As $\lambda \to \infty$, $\|u\|^4_{L^1[0,t_o]}$ tends to $(2\, t_o/\pi)^4$, while $\int_0^t (\int_0^s u_\lambda(\sigma)(d\sigma)^2 ds$ tends to zero. Bounds of the type $|\varepsilon_i(u,t)| < Ct^4$ or $|\varepsilon_i(u,t)| < C \cdot \|u\|^4$ are therefore too weak for proving non-controllability even in this simple case.

§6 Directional Estimates.

To obtain more precise bounds on the error in the Taylor expansion of (1.2), in this section we split $\mathbb{R}^d$ into a sum of orthogonal subspaces V_p and estimate the size of the error separately on each V_p Given the control system (1.2), define an increasing sequence of subspaces $S_p \subseteq \mathbb{R}^d$ recursively by setting i) $S_o = \{0\}$, ii) S_p is the smallest subspace of $\mathbb{R}^d$ such that for all $i = 1, \ldots, m$ and $k = 0, \ldots, p$ one has $D^k g_i(o)(z_1,\ldots,z_k) \in S_p$ for every k-tuple $(z_1,\ldots,z_k)$ with $z_i \in S_{j_i}$ and $j_1 + j_2 + \ldots + j_k < p$. In particular, S_1 is the smallest subspace that contains the m vectors $g_i(o)$ and is invariant under the linear operators $\nabla g_i(o)$ $(i = 1, \ldots, m)$. Now choose any $\bar{p} > 1$. For $1 < p < \bar{p}$ define V_p as the orthogonal complement of S_{p-1} in S_p. Let $V_{\bar{p}}$ be the orthogonal complement of $S_{\bar{p}-1}$ in $\mathbb{R}^d$. Finally, denote by π_p the orthogonal projection $\mathbb{R}^d \to V_p$. With this notation, we have

Theorem 2. Let $u(\cdot) \to x(u,\cdot)$ be the input-output map generated by (1.2) and let $u(\cdot) \to x_n(u,\cdot)$ be its n-th order Taylor approximation about the null control. If $p \in \{1,\ldots,\bar{p}\}$ and $n > p$, then there exist C_o, $t_o > 0$ such that

$$|\pi_p(x(u,t) - x_n(u,t))| < C_o t^{n-p} \int_0^t | \int_0^s G(o)u(\sigma)d\sigma|^p ds \tag{6.1}$$

for all $t \in [0,t_o]$, $u \in U$.

Proof. Let y_n be the solution of (5.3). By Lemma 2, the difference $y_n(u,t) - x(u,t)$ satistifies a bound of the form (6.1) for any $p = 1, \ldots, \bar{p}$. Therefore we only need to show that

$$|\pi_p(y_n(u,t) - x_n(u,t))| < C_o\, t^{n-p} \int_o^t |x_1(u,s)|^p ds,$$

where

$$x_1(u,s) = \int_o^s G(o)u(\sigma)d\sigma$$

gives the first order Taylor approximation to the trajectory $x(\cdot,\cdot)$. In §3, x_n was proven to be a finite sum of integral monomials occuring within the first n Picard iterates for the system (5.3). To keep track of its size, to each one of the above integromials we attach an integer $\gamma(\mu)$ as follows: If $\mu = x_1$, defined at (6.3), set $\gamma(\mu) = 1$. If

$$\mu(u,t) = \int_o^t \frac{1}{k!} G^{(k)}(o)(\mu_1(u,s),\ldots,\mu_k(u,s))u(s)ds, \tag{6.4}$$

set $\gamma(\mu) = \gamma(\mu_1) + \cdots + \gamma(\mu_k)$. We stress that this definition refers exclusively to the integromials arising via Picard iterations for the particular system (5.3) presently considered. Comparing the definitions of γ and of the subspaces S_p, it is clear that $\mu(u,t) \in S_{\gamma(\mu)}$ for all $t > 0$, $u \in U$. Also notice that if the order of μ is $\nu > 1$, then $\gamma(\mu) < \nu - 1$. A basic estimate on the size of integral monomials is now given.

Lemma 3. Let μ be an integral monomial of order $\nu > 1$ occurring in some Picard iterate for (5.3), and let $\gamma(\mu) = p$. Then there exists a constant C such that

$$|\mu(u,t)| < C\, t^{\nu-p-1} \int_o^t |x_1(u,s)|^p ds \tag{6.5}$$

for all $u \in U$, $0 < t < 1$.

<u>Proof</u>. Notice first that for the integromial $x_1(\cdot,\cdot)$ we have $\nu = p = 1$. For all others, $\nu > 1$. Moreover, the only integral monomials for which $\nu = p + 1$ have the form

$$\mu(u,t) = \int_o^t \frac{1}{p!} G^{(p)}(o)\ (x_1(u,s))^{[p]}u(s)\ ds \tag{6.6}$$

and clearly satisfy an estimate of the type (6.5). The general case will be proved by induction on ν, assuming $\nu > p + 2$. If μ is given by (6.4), let μ_i have order $\nu_i < \nu$ and let $\gamma(\mu_i) = p_i$ $(i = 1,\ldots,k)$. The inductive hypothesis implies that either $\mu_i = x_1$ or

$$|\mu_i(u,s)| < C_i \cdot s^{\nu_i - p_i - 1} \int_o^s |x_1(u,\sigma)|^{p_i} d\sigma \tag{6.7}$$

for some constant C_1 and all $u \in U$, $0 < t < 1$.

Using the symmetry of $G^{(k)}(o)$, we can reorder the μ_i and assume that (6.4) has the form

$$\mu(u,t) = \int_o^t \frac{1}{k!} G^{(k)}(o)(\mu_1(u,s),\ldots,\mu_h(u,s),\ x_1(u,s),\ldots,x_1(u,s))u(s)ds \qquad (6.8)$$

for some $0 < h < k$, with each $\mu_i (1 < i < h)$ having order $\nu_i > 1$. The order of μ in (6.8) is then $\nu = 1 + \nu_1 + \cdots + \nu_h + k - h$, while $p = \gamma(\mu) = p_1 + \cdots p_h + k - h$. In the following, we set $q = p_1 + \cdots + p_h = p - k + h$. If $q = 0$, then μ has the form (6.6) and the estimate (6.5) is immediate. If $q > 0$, the inductive hypothesis and Hölder's inequality imply

$$|\mu(u,t)| < C' \int_o^t |x_1(u,s)|^{k-h} \prod_{i=1}^{h} \left[C_i s^{\nu_i - p_i - 1} \int_o^s |x_1(u,\sigma)|^{p_i} d\sigma\right] ds$$

$$< C \int_o^t s^{(\nu-1)-p-h} \prod_{i=1}^{h} \left[\left(\int_o^s |x_1(u,\sigma)|^q d\sigma\right)^{p_i/q} \cdot s^{(q-p_i)/q} \right] ds$$

$$< C \int_o^t s^{\nu-p-2} |x_1(u,s)|^{k-h} \left(\int_o^s |x_1(u,\sigma)|^q d\sigma\right) ds, \qquad (6.9)$$

where the constants C, C', C_i are independent of u and t. If $h = k$, (6.5) is a trivial consequence of (6.9). If $h < k$, one recovers again (6.5) from (6.9) integrating by parts and using Hölder's inequality:

$$|\mu(u,t)| < C\, t^{\nu-p-2} \left(\int_o^t |x_1(u,s)|^{k-h} ds\right)\left(\int_o^t |x_1(u,s)|^q ds\right)$$

$$< C\, t^{\nu-p-1} \int_o^t |x_1(u,s)|^{k-h+q} ds.$$

Returning to the proof of Theorem 2, for any control $u \in U$ and any constants $C, \tau > 0$, define $\sum_u(C,\tau)$ as the set of all maps $z \in C([0,\infty); \mathbb{R}^d)$ such that

$$|\pi_p(z(t) - x_n(u,t))| < C \cdot t^{n-p} \int_o^t |x_1(u,s)|^p ds \qquad (6.10)$$

for all $t \in [0,\tau]$, $p = 1, \ldots, \bar{p}$. We claim that $y_n(u,\cdot) \in \sum_u(C_o,t_o)$ for some C_o, $t_o > 0$ and all $u \in U$.

Lemma 4. There exist constants C_o, t_o such that, for every $u \in U$, the map ψ_u defined by

$$\psi_u(z)(t) = \int_o^t G_{n-1}(z(s))u(s)ds$$

maps $\sum_u(C_o,t_o)$ into itself.

Proof. Given $u \in U$, - $C([o,\infty); \mathbb{R}^d)$, for $p = 1, \ldots, \bar{p}$ set

$w_p(t) = \pi_p(z(t) - x_n(u,t))$. We then have

$$|\pi_p(x_n)(u,t) - \psi_u(z(t)| \le |\pi_p(x_n(u,t) - \int_0^t G_{n-1}(x_n(u,s))u(s)ds)|$$

$$+ |\pi_p(\int_0^t G_{n-1}(x_n(u,s))u(s)ds - \int_0^t G_{n-1}(x_n(u,s) + \sum_{p=1}^{\bar{p}} w_p(s))u(s)ds)|$$

$$= |A(u,t)| + |B(u,t)|. \tag{6.11}$$

Notice that $A(u,t)$ is a finite sum of integral monomials which, by Corollary 1 in §2, have order $> n$. If μ is one of such monomials, then either $\gamma(\mu) > p$ or $\pi_p(\mu(u,t)) \equiv 0$. By Lemma 3, there exists a constant $C_1 > 0$ such that

$$|\pi_p(A(u,t))| \le C_1 t^{n-p} \int_0^t |x_1(u,s)|^p ds \tag{6.12}$$

for all $u \in U$, $0 \le t \le 1$, $p = 1, \ldots, \bar{p}$. Take $C_0 = 2C_1$. To determine a suitable t_0, observe that $B(u,t)$ can be written as a finite sum of terms of the type

$$\Lambda(u,t) = \int_0^t \frac{1}{k!} G^{(k)}(0)\ (w_{p_1}(s), \ldots, w_{p_\ell}(s), \mu_{\ell+1}(u,s), \ldots, \mu_{\ell+h}(u,s),$$

$$x_1(u,s), \ldots, x_1(u,s))u(s)ds,$$

where $k \in \{1,\ldots,n-1\}$, $\ell \ge 1$ and the integromials $\mu_{\ell+1}, \ldots, \mu_{\ell+h}$ have order > 1. For $i = \ell + 1, \ldots, \ell + h$, let $p_i = \gamma(\mu_i)$. Observe that the order of μ_i is then at least $p_i + 1$. If $z \in \sum_u(C_0, \tau)$ we thus have the inequalities

$$|\mu_i(u,s)| \le C \int_0^s |x_1(u,\sigma)|^{p_i} d\sigma,$$

$$|w_{p_i}(u,s)| \le C_0\, s^{n-p_i} \int_0^s |x_1(u,\sigma)|^{p_i} d\sigma,$$

for a suitable constant C and $s \le \min\{\tau, 1\}$. The definition of S_p implies that either $p_1 + \ldots + p_{\ell+h} + k - \ell - h > p$, or $\pi_p(\Lambda(u,t)) \equiv 0$. The same arguments used in the proof of Lemma 3 now yield constants $C_2, t_2 > 0$ such that

$$\pi_p(\Lambda(u,t)) \le C_2\, t^{n-p+1} \int_0^t |x_1(u,s)|^p ds, \tag{6.13}$$

for $0 < t < t_2$, $u \in U$.

Therefore, there exist $C_3, t_3 > 0$ such that

$$|\pi_p(B(u,t))| < C_3\, t^{n-p+1} \int_o^t |x_1(u,s)|^p ds \qquad (6.14)$$

for all $u \in U$, $p = 1, \ldots, \bar{p}$, $t \in [0,t_3]$. Comparing (6.11) with (6.12) and (6.14), we see that Lemma 4 holds with $t_o = \min \{1, t_3, C_1 C_3^{-1}\}$.

The conclusion of the proof of Theorem 2 is now straightforward. For all $u \in U$, $y_n(u,\cdot)$ is the unique fixed point of ψ_u. By Lemma 4, $y_n(u,\cdot) \in \sum_u(c_o,t_o)$, hence (6.2) follows from (6.10).

With the same notation of Theorem 2 we have

Corollary 3. If $p > 1$, $n < p$, then there exist $C_o, t_o > 0$ such that

$$|\pi_p(x(u,t) - x_n(u,t))| = |\pi_p(x(u,t))| < C_o \int_o^t | \int_o^s G(o)u(\sigma)d\sigma|^p ds \qquad (6.15)$$

for all $u \in U$, $t \in [0,t_o]$, $p = 1, \ldots, \bar{p}$.

Indeed, x_n is a sum of integral monomials μ_i having order $< n$. Hence $\gamma(\mu_i) < p$ and $\pi_p(\mu_i(u,t)) \equiv 0$. This implies $\pi_p(x_n(u,t)) \equiv 0$. Setting $n = p$, Theorem 2 yields the bound (6.15).

Example 4. *consider on* $\mathbb{R}^3$ the system $\dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3) = (u \cos x_1 - x_2 - x_3,\ tg(x_1 - x_3),\ \sin^2 x_1 - x_2^2 + x_1^3)$, $x(0) = 0 \in \mathbb{R}^3$. A third order expansion yields

$$x_1(u,t) = \int_o^t u(s)ds - \int_o^t \int_o^{\sigma_1} \int_o^{\sigma_2} u(\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1 + \varepsilon_1(u,t),$$

$$x_2(u,t) = \int_o^t \int_o^{\sigma_1} u(\sigma_2)d\sigma_2 d\sigma_1 + \varepsilon_2(u,t),$$

$$x_3(u,t) = \int_o^t \left(\int_o^s u(\sigma)d\sigma\right)^2 ds + \varepsilon_3(u,t).$$

For this system, $S_1 = V_1 =$
$\{(\xi_1, \xi_2, 0) ; \xi_1, \xi_2 \in \mathbb{R}\}$, $S_2 = \mathbb{R}^3$, $V_2 = \{(0,0,\xi_3) ; \xi_3 \in \mathbb{R}\}$. By Theorem 2 there exist $C, T > 0$ such that

$$| \varepsilon_3(u,t) | \le C\, t \int_0^t | \int_0^s u(\sigma)d\sigma |^2 ds$$

for all $u \in U$, $t \in [0,T]$. Hence, for small t, $x_3(u,t) \ge 0$ and the system is not locally controllable. An alternative proof of this could be obtained from the results in [6].

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Let g_i $(i = 1, \ldots, m)$ be smooth vector fields on $\mathbb{R}^d$, and let $T^{n-1} g_i$ be their Taylor expansions of order $n - 1$ at the origin. The system $\dot{x}(t) = \sum_{i=1}^{m} g_i(x(t))u_i(t)$, $x(o) = 0 \in \mathbb{R}^d$ generates an input-output map $u(\cdot) \to x(u,\cdot)$ whose n-th order Taylor approximation $x_n(u,\cdot)$ can be

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ABSTRACT (Continued)

obtained by computing the n-th Picard iterate for the reduced system $\dot{x}(t) = \sum_{i=1}^{m} (T^n g_i)(x(t))\, u_i(t)$, $x(o) = 0$, discarding the terms of order $> n$. For $z \in R^d$, directional error bounds of the form

$$| < z, x(u,t) - x_n(u,t) > | < C\, t^{n-p} \int_o^t \left| \int_o^s \sum_{i=1}^{m} g_i(o) u_i(\sigma)\, d\sigma \right|^p ds$$

can be given. These estimates improve those supplied by the classical Taylor's theorem and yield results concerning local non-controllability.